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Approximate linear response for slow variables of dynamics with explicit time scale separation

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ABSTRACT

Many real-world numerical models are notorious for the time scale separation of different subsets of variables and the inclusion of random processes. The existing algorithms of linear response to external forcing are vulnerable to the time scale separation due to increased response errors at fast scales. Here we develop the approximate linear response algorithm for slow variables in a two-scale dynamical system with explicit separation of slow and fast variables, which has improved numerical stability and reduced computational expense.

1. Introduction

Recently, Majda and the author [1–4] developed and tested a novel computational algorithm for predicting the mean response of nonlinear functions of states of a chaotic dynamical system to small change in external forcing via the fluctuation–dissipation theorem (FDT). This algorithm (called the short-time FDT (ST-FDT) algorithm in [2–4]) takes into account the fact that the dynamics of chaotic nonlinear forced-dissipative systems often reside on chaotic fractal attractors, where the classical quasi-Gaussian formula of the fluctuation–dissipation theorem often fails to produce satisfactory response prediction, especially in dynamical regimes with weak and moderate chaos and slower mixing. It has been discovered that the ST-FDT algorithm is an extremely precise response approximation for short response times, and can be blended with the classical quasi-Gaussian FDT algorithm (qG-FDT) for longer response times to alleviate negative effects of expanding Lyapunov directions. Additionally, in [1] the author developed a computationally inexpensive approximate method for ST-FDT using the reduced-rank tangent map. Majda and Wang [9] developed a comprehensive linear response framework in the case of non-autonomous dynamics with time-periodic forcing (which also applies for general non-autonomous dynamics).

However, in multiscale dynamical systems with time scale separation the ST-FDT method can be vulnerable to the presence of the fast variables, especially when the response is practically needed only for slow model variables (such as those in a climate system), due to increased response errors at fast scales. Moreover, it is often the case that there are only a few slow variables in the model and a large number of fast variables. Even if only the response of the slow variables is needed, in a straightforward implementation such as that in [3,2,4], the ST-FDT response operator has to be computed for all variables in the model, which can be computationally expensive or even practically impossible for models with large sets of fast variables.

In the work, we develop an approximate response algorithm based on averaged dynamics of multiscale systems [10,12,13]. The new method allows to compute the response operators directly at slow variables using existing FDT formulas, which improves numerical stability and reduces computational expense.

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2. Average response for two-scale systems perturbed at slow variables

Consider a two-scale system of Itô stochastic differential equations of the form

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y},t),\\ \mathrm{d}\boldsymbol{y} &= \frac{1}{\varepsilon}\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y},t,\varepsilon)\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{y},t,\varepsilon)\mathrm{d}\boldsymbol{W}_t,\end{aligned} \tag{2.1}$$

where $\boldsymbol{x} = \boldsymbol{x}(t) \in \mathbb{R}^{N_x}$ is the set of slow variables, and $\boldsymbol{y} = \boldsymbol{y}(t) \in \mathbb{R}^{N_y}$ is the set of fast variables. This situation is common in geophysical science, where the time scale of different state variables of a weather/climate system can range between minutes and months (or even years), and fast variables are often driven by a random process. In (2.1) we use the following notations: $\boldsymbol{W}_t$ is the K-dimensional Wiener process, $\boldsymbol{F}$ and $\boldsymbol{G}$ are N_x and N_y vector-valued functions of $\boldsymbol{x}$, $\boldsymbol{y}$ and t, and $\boldsymbol{\sigma}$ is a $N_y \times K$ matrix-valued function of $\boldsymbol{x}$, $\boldsymbol{y}$ and t. For the purpose of this work, here we assume that there is a constant parameter $0 < \varepsilon \ll 1$ which sets the time scale separation between $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ into slow and fast variables, respectively, and we additionally assume that $\boldsymbol{G}$ and $\boldsymbol{\sigma}$ continuously depend on ε.

We make a general assumption that, given a time t_0 and a probability measure $\mathcal{P}_{t_0}$, the non-autonomous dynamical system in (2.1) transports it into $\mathcal{P}_{t_0+t}$, where t is the elapsed interval of time after t_0. Then, by ρ_{t_0+t} we denote the marginal measure of $\mathcal{P}_{t_0+t}$ for the set of slow variables $\boldsymbol{x}$, such that for any observable $A(\boldsymbol{x})$ its average value $\langle A\rangle(t_0+t)$ is given by

$$\langle A\rangle(t_0+t) = \rho_{t_0+t}(A) \equiv \int_{\mathbb{R}^{N_x}} A(\boldsymbol{x})\mathrm{d}\rho_{t_0+t}(\boldsymbol{x}). \tag{2.2}$$

We say that the system in (2.1) is *perturbed at slow variables* when there is a small forcing $\boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t)$ applied at slow variables:

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y},t) + \boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t),\\ \mathrm{d}\boldsymbol{y} &= \frac{1}{\varepsilon}\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y},t,\varepsilon)\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{y},t,\varepsilon)\mathrm{d}\boldsymbol{W}_t,\end{aligned} \tag{2.3}$$

where $\boldsymbol{w}(\boldsymbol{x})$ is an $N_x \times L$ matrix-valued function of $\boldsymbol{x}$, while $\delta\boldsymbol{f}(t)$ is a L vector-valued function of time t for some integer L. For the perturbed system in (2.3) we assume that $\mathcal{P}_{t_0}$ is transported into $\mathcal{P}^*_{t_0+t}$ with the corresponding marginal measure $\rho^*_{t_0+t}$ for slow variables. Finally, we define the *average response* of $A(\boldsymbol{x})$ to the small forcing in (2.3), starting at t_0, as

$$\delta\rho_{t_0+t}(A) = \rho^*_{t_0+t}(A) - \rho_{t_0+t}(A). \tag{2.4}$$

Our goal here is to compute a linearization of (2.4) with respect to the forcing $\boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t)$ in (2.3) under the assumption that both the forcing and the response are small, which is provided by FDT [1,3,2,4,8]. Observe that both the forcing $\boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t)$ and the response function $A(\boldsymbol{x})$ involve only slow variables $\boldsymbol{x}$. However, a straightforward application of the FDT linearization of the response to (2.4) will lead to the computation of the linear response operator for the complete set of model variables, that is, for both $\boldsymbol{x}$ and $\boldsymbol{y}$. This is undesirable for the following reasons: first, it can make the computation of the response expensive (especially if there are many fast variables, which is often the case); and, second, the stability of the ST-FDT response operator may suffer due to large Lyapunov exponents at fast variables. In what follows we develop the approximate FDT formulas under the assumption that the behavior of the slow variables in (2.1) approaches the limiting case of "infinitely fast" $\boldsymbol{y}$-variables.

3. Limiting dynamics for slow variables

We rescale the time in (2.1) and (2.3) as $t = \varepsilon\tilde{t}$. For the rescaled time $\tilde{t}$, the equation for fast variables $\boldsymbol{y}$ in both (2.1) and (2.3) becomes

$$\mathrm{d}\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{x},\boldsymbol{y},\varepsilon\tilde{t},\varepsilon)\mathrm{d}\tilde{t} + \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{y},\varepsilon\tilde{t},\varepsilon)\mathrm{d}\boldsymbol{W}_{\tilde{t}}. \tag{3.1}$$

Following [10,12,13], in the vicinity of some $\boldsymbol{x}$ and t, we write the separate degenerate system for the fast variables in the limiting form as $\varepsilon \to 0$:

$$\mathrm{d}\boldsymbol{z} = \boldsymbol{G}(\boldsymbol{x},\boldsymbol{z},t,0)\mathrm{d}\tilde{t} + \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{z},t,0)\mathrm{d}\boldsymbol{W}_{\tilde{t}}, \tag{3.2}$$

where $\boldsymbol{x}$ and t are treated as constant parameters, and, therefore, $\boldsymbol{z}$ parametrically depends on $\boldsymbol{x}$ and t. Observe that the limiting system for the fast variables in (3.2) is autonomous, that is, it does not explicitly depend on $\tilde{t}$ (except for the Wiener process). Here we assume that (3.2) possesses the invariant ergodic probability measure $\mu_{\boldsymbol{x},t}$, which depends on $\boldsymbol{x}$ and t as parameters.

Now, following [10,12,13] we write the *averaged* perturbed and unperturbed systems for the slow variables $\boldsymbol{x}$ as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \overline{\boldsymbol{F}}(\boldsymbol{x},t), \tag{3.3}$$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \overline{\boldsymbol{F}}(\boldsymbol{x},t) + \boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t), \tag{3.4}$$

respectively, with $\overline{\boldsymbol{F}}$ defined as

$$\overline{\boldsymbol{F}}(\boldsymbol{x},t) = \int_{\mathbb{R}^{N_y}} \boldsymbol{F}(\boldsymbol{x},\boldsymbol{z},t)\mathrm{d}\mu_{\boldsymbol{x},t}(\boldsymbol{z}). \tag{3.5}$$

Observe that both (3.3) and (3.4) are fully deterministic. For identical initial conditions $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ the solutions of (3.3) and (3.4) constitute approximations to the solutions of the equations for slow variables in (2.1) and (2.3), respectively, for a finite interval of time $t \sim 1$ [10,12,13]. We let (3.3) and (3.4) generate the flows $\bar{\phi}^{t_0,t}$ and $\bar{\phi}^{*t_0,t}$, respectively, such that, for $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, $\bar{\phi}^{t_0,t}\boldsymbol{x}_0$ corresponds to the solution of the averaged equation in (3.3), and $\bar{\phi}^{*t_0,t}\boldsymbol{x}_0$ corresponds to the solution of the perturbed averaged equation in (3.4).

4. Approximate fluctuation–dissipation theorem for slow variables

Observe that, for $t \sim 1$, $\rho_{t_0+t}(A) \approx \rho_{t_0}(A \circ \bar{\phi}^{t_0,t})$, and $\rho^*_{t_0+t}(A) \approx \rho_{t_0}(A \circ \bar{\phi}^{*t_0,t})$, since the equations for characteristics (3.3) and (3.4), along which the measure is transported, are the approximations of (2.1) and (2.3), respectively. With this, we now define the approximate average response of $A(\boldsymbol{x})$ to the small forcing in (2.3), starting at time t_0, as

$$\bar{\delta}\rho_{t_0+t}(A) = \int_{\mathbb{R}^{N_x}} \left[A(\bar{\phi}^{*t_0,t}\boldsymbol{x}) - A(\bar{\phi}^{t_0,t}\boldsymbol{x})\right]\mathrm{d}\rho_{t_0}(\boldsymbol{x}), \tag{4.1}$$

where $\bar{\phi}^{t_0,t}$ and $\bar{\phi}^{*t_0,t}$ are the flows generated by (3.3) and (3.4), respectively. Upon linearization, the response is given by

$$\bar{\delta}\rho_{t_0+t}(A) = \int DA(\bar{\phi}^{t_0,t}\boldsymbol{x})\delta\bar{\phi}^{t_0,t}\boldsymbol{x}\,\mathrm{d}\rho_{t_0}(\boldsymbol{x}), \tag{4.2}$$

with $\delta\bar{\phi}^{t_0,t}\boldsymbol{x}$ being the solution of

$$\frac{\partial}{\partial t}\delta\bar{\phi}^{t_0,t}\boldsymbol{x} = D_{\boldsymbol{x}}\overline{\boldsymbol{F}}(\bar{\phi}^{t_0,t}\boldsymbol{x},t_0+t)\delta\bar{\phi}^{t_0,t}\boldsymbol{x} + \boldsymbol{w}(\bar{\phi}^{t_0,t}\boldsymbol{x})\delta\boldsymbol{f}(t_0+t), \quad \delta\bar{\phi}^{t_0,0}\boldsymbol{x} = 0, \tag{4.3}$$

where $D_{\boldsymbol{x}}\overline{\boldsymbol{F}}$ is the Jacobian of $\overline{\boldsymbol{F}}$ from (3.3) and (3.4) [1,11]. The solution to (4.3) is given by

$$\delta\bar{\phi}^{t_0,t}\boldsymbol{x} = \int_0^t \overline{\boldsymbol{T}}^{t_0+\tau,t-\tau}_{\bar{\phi}^{t_0,\tau}\boldsymbol{x}}\boldsymbol{w}(\bar{\phi}^{t_0,\tau}\boldsymbol{x})\delta\boldsymbol{f}(t_0+\tau)\mathrm{d}\tau, \tag{4.4}$$

where $\overline{\boldsymbol{T}}$ is the tangent map of $\bar{\phi}$, computed by solving

$$\frac{\partial}{\partial t}\overline{\boldsymbol{T}}^{t_0,t}_{\boldsymbol{x}} = D_{\boldsymbol{x}}\overline{\boldsymbol{F}}(\bar{\phi}^{t_0,t}\boldsymbol{x},t_0+t)\overline{\boldsymbol{T}}^{t_0,t}_{\boldsymbol{x}}, \quad \overline{\boldsymbol{T}}^{t_0,0}_{\boldsymbol{x}} = I. \tag{4.5}$$

With (4.4), the linear response formula in (4.2) can be written as

$$\bar{\delta}\rho_{t_0+t}(A) = \int_0^t \overline{\boldsymbol{R}}_{ST}(t_0,t,\tau)\delta\boldsymbol{f}(t_0+\tau)\mathrm{d}\tau, \tag{4.6}$$

where the averaged short-time linear response operator (AST-FDT) $\overline{\boldsymbol{R}}_{ST}(t_0,t,\tau)$ is given by

$$\overline{\boldsymbol{R}}_{ST}(t_0,t,\tau) = \int DA(\bar{\phi}^{t_0,t}\boldsymbol{x})\overline{\boldsymbol{T}}^{t_0+\tau,t-\tau}_{\bar{\phi}^{t_0,\tau}\boldsymbol{x}}\boldsymbol{w}(\bar{\phi}^{t_0,\tau}\boldsymbol{x})\mathrm{d}\rho_{t_0}(\boldsymbol{x}). \tag{4.7}$$

If the probability measure ρ_{t_0} is absolutely continuous with respect to the Lebesgue's measure for any t_0 (which is usually the case for stochastically driven fast variables in (2.1)), i.e. $\mathrm{d}\rho_{t_0}(\boldsymbol{x}) = p_{t_0}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ with $p_{t_0}(\boldsymbol{x})$ being a known smooth probability density function, then, using the approximation $\rho_{t_0}(A \circ \bar{\phi}^{t_0,t}) \approx \rho_{t_0+t}(A)$ for $t \sim 1$, one can integrate the formula in (4.7) by parts, obtaining the classical FDT formula

$$\overline{\boldsymbol{R}}_{class}(t_0,t,\tau) = -\int A(\bar{\phi}^{t_0+\tau,t-\tau}\boldsymbol{x})\mathrm{div}(\boldsymbol{w}(\boldsymbol{x})p_{t_0+\tau}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}. \tag{4.8}$$

5. Restrictions on $\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{\sigma}$ for practical computation

As shown above, the computation of the ST-FDT response formula from (3.3) requires explicit knowledge of the Jacobian of $\overline{\boldsymbol{F}}$, which necessitates either explicit knowledge of $\mu_{\boldsymbol{x},t}$ for (3.2) or its independence of $\boldsymbol{x}$. Here we operate under the assumption that $\mu_{\boldsymbol{x},t}$ is generally unknown, which requires its independence on $\boldsymbol{x}$:

$$\mu_{\boldsymbol{x},t} = \mu_t. \tag{5.1}$$

In order to satisfy (5.1), we have to require for (2.1) and (2.3) that

$$\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y},t,0) = \hat{\boldsymbol{G}}(\boldsymbol{y},t), \quad \boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{y},t,0) = \hat{\boldsymbol{\sigma}}(\boldsymbol{y},t), \tag{5.2}$$

that is, when ε in $\boldsymbol{G}$ and $\boldsymbol{\sigma}$ is set to zero, the dependence on $\boldsymbol{x}$ vanishes. With (5.1) and (5.2), (3.5) becomes

$$\overline{\boldsymbol{F}}(\boldsymbol{x},t) = \int_{\mathbb{R}^{N_y}} \boldsymbol{F}(\boldsymbol{x},\boldsymbol{z},t)\mathrm{d}\mu_t(\boldsymbol{z}). \tag{5.3}$$

Here one can observe that, even with restrictions on $\boldsymbol{G}$ and $\boldsymbol{\sigma}$ in (5.2), it is still not possible to compute the Jacobian of $\overline{\boldsymbol{F}}$ for a general $\boldsymbol{F}$ and unknown μ_t, and, therefore, additional restrictions must be placed on $\boldsymbol{F}$. Although more general restrictions are possible, here, for simplicity, we restrict $\boldsymbol{F}$ to the form

$$\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y},t) = \boldsymbol{f}(\boldsymbol{x},t) + \boldsymbol{h}(\boldsymbol{y},t), \tag{5.4}$$

which is frequent in geophysical science applications, with

$$\overline{\boldsymbol{F}}(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x},t) + \bar{\boldsymbol{h}}(t), \quad \bar{\boldsymbol{h}}(t) = \int_{\mathbb{R}^{N_y}} \boldsymbol{h}(\boldsymbol{z},t)\mathrm{d}\mu_t(\boldsymbol{z}). \tag{5.5}$$

Then, the Jacobian of $\overline{\boldsymbol{F}}$ is the same as the Jacobian of $\boldsymbol{f}$, that is,

$$D_{\boldsymbol{x}}\overline{\boldsymbol{F}}(\boldsymbol{x},t) = D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x},t). \tag{5.6}$$

6. Approximate FDT for autonomous dynamics

At this point, for practical computation of the linear response operators we need to convert the formulas in (4.7) and (4.8) from measure averages to time averages over a long-time trajectory of (2.1). For the special case of autonomous dynamics (i.e. without explicit time dependence in (2.1)) we have

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}),\\ \mathrm{d}\boldsymbol{y} &= \frac{1}{\varepsilon}\boldsymbol{G}(\boldsymbol{x},\boldsymbol{y},\varepsilon)\mathrm{d}t + \frac{1}{\sqrt{\varepsilon}}\boldsymbol{\sigma}(\boldsymbol{x},\boldsymbol{y},\varepsilon)\mathrm{d}\boldsymbol{W}_t,\end{aligned} \tag{6.1}$$

with the corresponding averaged system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \overline{\boldsymbol{F}}(\boldsymbol{x}). \tag{6.2}$$

The situation we consider here is that initially the system in (6.1) is placed in its statistical equilibrium (that is, $\rho_{t_0} = \rho$ is the marginal for slow variables $\boldsymbol{x}$ of the invariant measure for (6.1)), and then forced out of its statistical equilibrium by an external perturbation $\boldsymbol{w}(\boldsymbol{x})\delta\boldsymbol{f}(t)$. For this setup, we obtain

$$\begin{aligned}\bar{\delta}\rho_t(A) &= \int_0^t \overline{\boldsymbol{R}}_{ST}(t-\tau)\delta\boldsymbol{f}(t_0+\tau),\\ \overline{\boldsymbol{R}}_{ST}(t) &= \lim_{s\to\infty}\frac{1}{S}\int_0^s DA(\boldsymbol{x}(t+\tau))\overline{\boldsymbol{T}}^t_{\boldsymbol{x}(\tau)}\boldsymbol{w}(\boldsymbol{x}(\tau))\mathrm{d}\tau,\\ \frac{\partial}{\partial t}\overline{\boldsymbol{T}}^t_{\boldsymbol{x}} &= D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}(t))\overline{\boldsymbol{T}}^t_{\boldsymbol{x}}, \quad \overline{\boldsymbol{T}}^0_{\boldsymbol{x}} = I,\end{aligned} \tag{6.3}$$

where $\boldsymbol{x}(t)$ is the long-time trajectory of the slow variables from (6.1). Above we use the fact that $\boldsymbol{x}(t+\tau) \approx \bar{\phi}^t\boldsymbol{x}(\tau)$ for finite time $t \sim 1$. Similarly, for the classical FDT, and $p_{t_0}(\boldsymbol{x}) = p(\boldsymbol{x})$ we obtain

$$\overline{\boldsymbol{R}}_{class}(t) = -\lim_{s\to\infty}\frac{1}{S}\int_0^s A(\boldsymbol{x}(t+\tau)) \times [\mathrm{div}(\boldsymbol{w}(\boldsymbol{x}(\tau)) + \boldsymbol{w}(\boldsymbol{x}(\tau))\nabla\log p(\boldsymbol{x}(\tau))]\mathrm{d}\tau. \tag{6.4}$$

Further, if the invariant measure has a Gaussian marginal probability density for the slow variables $\boldsymbol{x}$ with mean $\bar{\boldsymbol{x}}$ and covariance $\boldsymbol{C}$, then the classical FDT response formula can be written as a quasi-Gaussian FDT formula [3,2,4]:

$$\overline{\boldsymbol{R}}_{qG}(t) = \lim_{s\to\infty}\frac{1}{S}\int_0^s A(\boldsymbol{x}(t+\tau)) \times \left[-\mathrm{div}(\boldsymbol{w}(\boldsymbol{x}(\tau)) + \boldsymbol{w}(\boldsymbol{x}(\tau))\boldsymbol{C}^{-2}(\boldsymbol{x}(\tau)-\bar{\boldsymbol{x}})\right]\mathrm{d}\tau. \tag{6.5}$$

In practice, (6.3), (6.4) and (6.5) are computed as time autocorrelation functions along a single long-time trajectory of the unperturbed system in (6.1).

7. Special case with fully deterministic dynamics

Above we assumed that the limiting dynamics in (3.2) possess the ergodic invariant probability measure $\mu_{\boldsymbol{x},t}$ (or, for the assumptions in Section 5, μ_t). However, if the dynamics in (2.1) is fully deterministic ($\boldsymbol{\sigma} = 0$), then the invariant measure for (3.2) may no longer be ergodic (that is, the averaged dynamics of $\boldsymbol{x}$ in (3.2) may depend on the particular ergodic component of μ being sampled). However, for the restrictions on $\boldsymbol{F}$ in (5.4) we claim that it is not an issue: observe that the Jacobian of $\overline{\boldsymbol{F}}$

in (5.6), necessary to compute the tangent map in (6.3) remains the same (as only $\bar{\boldsymbol{h}}$ in (5.5) is affected by non-ergodicity of μ), while the actual trajectory of the underlying dynamics in (2.1) is used to propagate the tangent map forward in (6.3), such that μ_t is never practically involved. In order to support this claim, all the numerical experiments, performed in this work below, are carried out with a fully deterministic two-scale dynamical system.

8. Numerical setup and results

In this section we present numerical experiments of the new linear response algorithm for a model with an explicit time separation of variables into two scales.

Our choice of the testbed numerical model is the full Lorenz 96 (L96) model [5–7], given by

$$\begin{aligned} \dot{x}_k &= x_{k-1}(x_{k+1}-x_{k-2}) - dx_k + F - \lambda\sum_{j=1}^{J} y_{k,j}, \\ \dot{y}_{k,j} &= \frac{1}{\varepsilon}[y_{k,j+1}(y_{k,j-1}-y_{k,j+2}) - dy_{k,j} + F] + \lambda x_k, \\ &\varepsilon > 0, \quad \lambda > 0, \quad F = \text{const}, \end{aligned} \tag{8.1}$$

where $1 \leqslant k \leqslant N_x$, $1 \leqslant j \leqslant J$. The following notations are adopted above:

- $\boldsymbol{x}$ is a set of slow variables of size N_x. The following periodic boundary conditions hold for $\boldsymbol{x}$: $x_{k+N_x} = x_k$.
- $\boldsymbol{y}$ is a set of fast variables of size $N_y = N_x J$, where J is a positive integer. The following boundary conditions hold for $\boldsymbol{y}: y_{k+N_x,j} = y_{k,j}$ and $y_{k,j+J} = y_{k+1,j}$.
- F is the constant forcing parameter;
- d is the dissipation parameter, set to 1 for $F \neq 0$, and 0 for $F = 0$;
- ε is the time scale separation parameter;
- λ is the coupling parameter.

Originally in [5–7] there is no constant forcing F term in the equation for $\boldsymbol{x}$-variables in (8.1), however, in its absence the behavior of $\boldsymbol{y}$-variables is strongly dissipative [1], and here we add constant forcing F in the right-hand side of the second equation in (8.1) to induce strongly chaotic behavior of $\boldsymbol{y}$-variables with large positive Lyapunov exponents. Observe that the equations in (8.1) obey the restrictions on $\boldsymbol{F}$ and $\boldsymbol{G}$ from (5.2) and (5.4). The equations for the fast variables in (8.1) do not contain stochastic forcing as in (2.1) to demonstrate that the potential non-ergodicity of the invariant probability measure in (3.2) is not critical, as argued in Section 7.

In the case of zero F and d, the full L96 model in (8.1) preserves the quadratic energy of the form

$$E = \frac{1}{2}\sum_{k=1}^{N_x}\left(x_k^2 + \sum_{j=1}^{J} y_{k,j}^2\right), \tag{8.2}$$

and possesses the Liouville (incompressibility) property such that the equilibrium statistical state for (8.1) approaches the classical Gibbs equilibrium state with zero mean state and uniform energy spectrum as the number of variables tends to infinity. As a result, the classical FDT formula with Gaussian invariant probability measure (which we call the quasi-Gaussian FDT, or qG-FDT) is a good response approximation for the L96 model without forcing and dissipation.

Here we study the response of the mean state $\bar{\boldsymbol{x}}$ of the slow variables, such that $A(\boldsymbol{x}) = \boldsymbol{x}$, of the L96 model to a small constant external perturbation $\delta\boldsymbol{f} \in \mathbb{R}^{N_x}$, "switched on" at response time $t = 0$ (such that $\boldsymbol{w} = \boldsymbol{I}$). Under the above assumptions, the linear response for the slow variables of the full L96 model without forcing and dissipation is given by

$$\begin{aligned} &\delta\bar{\boldsymbol{x}} = \mathcal{R}(t)\delta\boldsymbol{f}, \quad \overline{\mathcal{R}}_{ST}(t) = \lim_{s\to\infty}\frac{1}{s}\int_0^s \mathrm{d}\tau\int_0^t \overline{\boldsymbol{T}}^r_{\boldsymbol{x}(\tau)}\mathrm{d}r, \\ &\overline{\mathcal{R}}_{QG}(t) = -\lim_{s\to\infty}\frac{1}{s}\int_0^s \mathrm{d}\tau\int_0^t \boldsymbol{x}(r+\tau)\boldsymbol{C}^{-2}(\boldsymbol{x}(\tau)-\bar{\boldsymbol{x}})\mathrm{d}r, \end{aligned} \tag{8.3}$$

where $\boldsymbol{C}^2$ is the statistical covariance matrix of the slow variables $\boldsymbol{x}$. We compare the results with the full ST-FDT linear response computed from the full set of variables $\boldsymbol{x} \times \boldsymbol{y}$, as it is normally done in [1,3,2,4], which is given by

$$\mathcal{R}_{ST}(t) = \lim_{s\to\infty}\frac{1}{s}\int_0^s \mathrm{d}\tau\int_0^t \boldsymbol{T}^r_{\boldsymbol{x}(\tau)}\mathrm{d}r, \tag{8.4}$$

where $\boldsymbol{T}^r_{\boldsymbol{x}}$ is the tangent map of the full L96 system in (8.1). Below we study the errors in response produced by the response operators $\mathcal{R}_{ST}$, $\overline{\mathcal{R}}_{ST}$ and $\mathcal{R}_{QG}$. The errors are determined by comparison with the full ideal response operator $\mathcal{R}_I$, which is computed by perturbing the model directly via a series of small perturbations [3,2,4,8]. The following parameters are used in the computation:

- $N_x = 8$, $N_y = 64$ (that is, $J = 8$, 72 variables in total);
- $\varepsilon = 0.1$ (weak time scale separation), $\varepsilon = 0.01$ (strong time scale separation);

- $\lambda = 0.1$ (weak coupling), $\lambda = 0.5$ (intermediate coupling), $\lambda = 1$ (strong coupling);
- Two dynamical regimes:
  - No forcing and dissipation, the time series are generated on a constant energy sphere of radius 1.
  - Forced-dissipative regime, $F = 6$, $d = 1$.

During the course of computations, the observed speed-up of AST-FDT over standard ST-FDT was about 200 times. In Fig. 1 we show the relative errors between the ideal response operator and the ST-FDT, AST-FDT, and qG-FDT response operators for the slow variables of the L96 model without forcing and dissipation for two values of the time scaling parameter ε, 0.01 and 0.1, and two values of the coupling parameter λ, 0.1 and 0.5. Observe that the error and blow-up time of the ST-FDT operator strongly depends of the value of ε: for $\varepsilon = 0.1$ the blow-up time is roughly 1.5 time units, while for $\varepsilon = 0.01$ it is about 0.2 time units. This happens due to the fact that the Lyapunov characteristic time for the full set of variables $X \times Y$ is roughly 10 times shorter for $\varepsilon = 0.01$ than that for $\varepsilon = 0.1$ (the fast variables are roughly ten times "faster" for $\varepsilon = 0.01$). The qG-FDT response operator is "exact" for the L96 model without forcing and dissipation as the equilibrium state of the model approaches the Gaussian distribution, and produces the smallest error among the computed FDT response operators. In the case of the weak coupling $\lambda = 0.1$ the AST-FDT operator produces comparable errors to the qG-FDT operator for both the weak ($\varepsilon = 0.1$) and strong ($\varepsilon = 0.01$) time scale separation. Remarkably, in the case of the intermediate coupling $\lambda = 0.5$ and weak time scale separation $\varepsilon = 0.1$ the AST-FDT operator produces significantly larger errors than the qG-FDT operator, but as the time scale separation becomes strong ($\varepsilon = 0.01$), the AST-FDT becomes roughly as precise as the qG-FDT operator.

In Fig. 2 we show the relative errors between the ideal response operator and the ST-FDT, AST-FDT, and qG-FDT response operators for the slow variables of the L96 model with $F = 6$ for two values of the time scaling parameter ε, 0.01 and 0.1, and three values of the coupling parameter λ, 0.1, 0.5 and 1. Again, observe that the error and blow-up time of the ST-FDT operator strongly depends of the value of ε: for $\varepsilon = 0.1$ the blow-up time is roughly 1.5 time units, while for $\varepsilon = 0.01$ it is about 0.2 time units. The qG-FDT response operator does not produce a good response approximation for all considered values of ε and λ, which is due to the fact that the equilibrium statistical state of the model is no longer Gaussian. On the other hand, for weak and intermediate coupling $\lambda = 0.1$, 0.5 the AST-FDT operator yields small errors for weak time scale separation

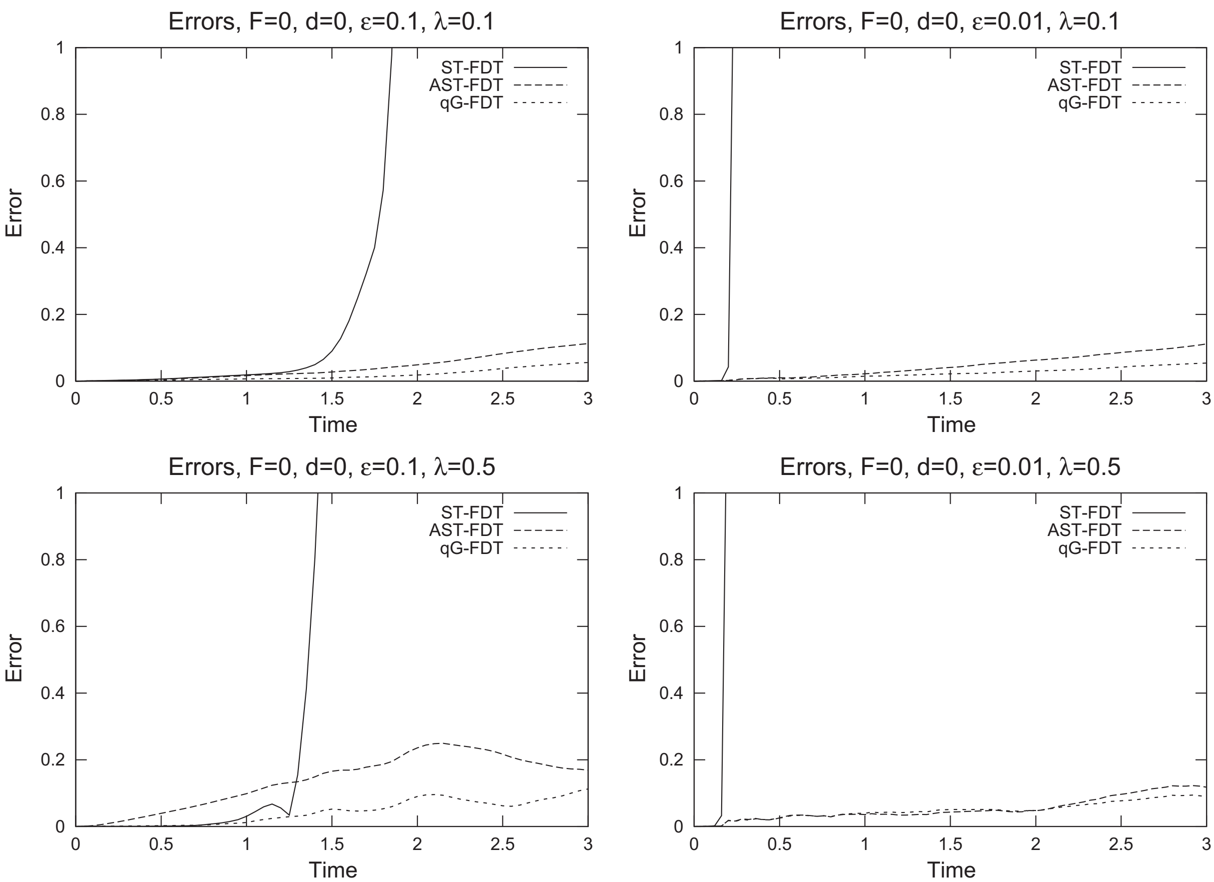

Fig. 1. The relative L_2 errors between the ideal and various FDT response operators for the L96 model without forcing and dissipation.

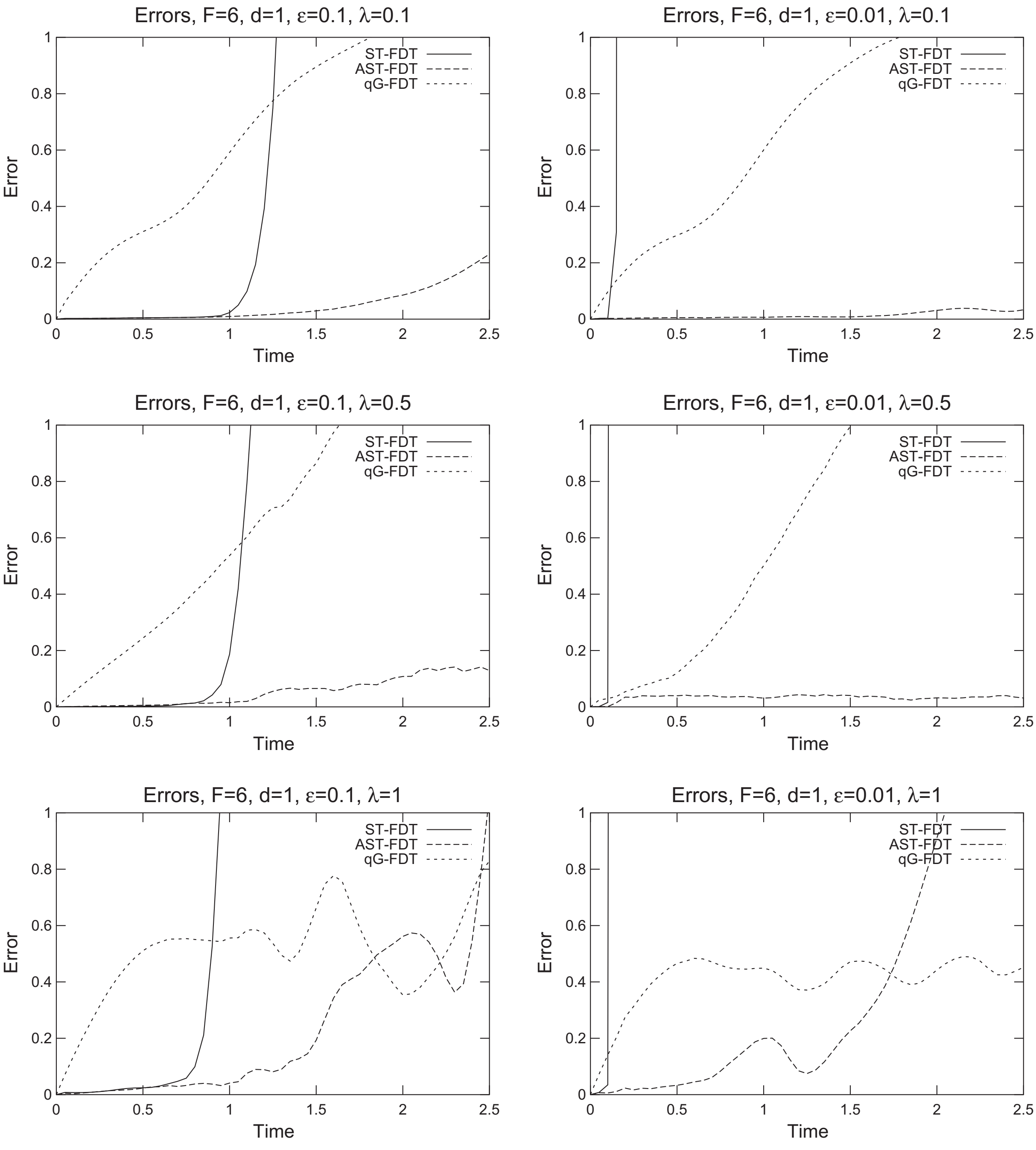

Fig. 2. The relative L_2 errors between the ideal and various FDT response operators for the L96 model with $F = 6$ and $d = 1$.

($\varepsilon = 0.1$) and further improves for strong time scale separation ($\varepsilon = 0.01$). For the strong coupling $\lambda = 1$, the AST-FDT operator again provides the smallest errors among all compared operators until the time $T = 1.8$, however later exhibits relatively early blow-up (as compared to $\lambda = 0.1, 0.5$ where no blow-up is observed).

9. Conclusions

In the work we developed a new response algorithm based on the approximate averaged dynamics of two-scale dynamical systems with optional stochastic forcing at fast variables. The new method allows to compute the response operators directly at slow variables using existing FDT formulas, improving numerical stability and reducing computational expense as compared to the existing methods. The new algorithm has the following practical advantages:

- By implementation, the algorithm is a direct application of the ST-FDT method separately onto the slow variables, which ensures easy practical implementation.
- There is a significant computational advantage in the case of small number of slow variables and large number of fast variables, since all the FDT formulas are computed for slow variables only (even though the long-term time series has to be computed from the full system in (2.1)).
- The new method does not appear to be sensitive to the presence of large Lyapunov exponents at fast variables, and the numerical stability of its computation is largely restricted by the characteristic Lyapunov time of the slow variables.

The new method is tested on the two-layer Lorenz 96 model with explicit time scale separation of slow and fast variables through a small parameter ε. The model is run in two regimes: one is without forcing and dissipation, where the quasi-Gaussian FDT provides a valid approximation; another regime is with forcing and dissipation, where the AST-FDT formula is necessary to calculate a good approximation to the linear response. In both cases, the new AST-FDT algorithm is observed to be superior to the standard ST-FDT in both numerical stability and computational expense, and more accurate than the qG-FDT for the regime with forcing and dissipation.

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